12th Week Friday

12 Mayıs 2023 Cuma 10:31

→ Eigenvalues & Eigenvectors

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

scalar

$A_{n\times n}$ → square matrix

For nonzero vectors $\vec{x}$,

$\lambda \in \mathbb{R}$

If the equation $\boxed{A\vec{x} = \lambda\vec{x}}$ has solution for nontrivial $\vec{x}$, we call λ as an "eigenvalue" of the square matrix A.

For a specific eigenvalue λ of A, solutions to $A\vec{x} = \lambda\vec{x}$ are called the "eigenvectors" corresponding to this λ.

$n\times 1$

basis vectors of the eigenspace

solution space $\leq \mathbb{R}^n$

eigenspace → basis

$$A\vec{x} = \lambda\vec{x}$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \underset{\text{scalar}}{\lambda} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$n\times 1$ $n\times 1$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(\underset{\text{matrix}}{A} - \underset{\text{scalar}}{\lambda} I)\vec{x} = \vec{0}$$

looking for possible solutions → eigenspace.

$n\times n$ identity matrix $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix}_{n\times n}$

$$\underbrace{\left(\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{n\times n} - \lambda \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n\times n} \right)}_{\text{square}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

→ a "homogeneous" system of linear equations

$$\underbrace{(A - \lambda I)}_{\text{square matrix}}\vec{x} = \vec{0}$$

→ we don't want only the trivial solution for $\vec{x}$

⇒ we want infinitely many solutions

$\det \neq 0$ $\xrightarrow{RREF}$ I_n $x_1 = 0$ $x_2 = 0$ $x_n = 0$ → trivial ✗

we want $\underline{\det = 0}$!

$$\boxed{\det(A - \lambda I) = 0}$$

→ In order to have nontrivial solutions this equality should be satisfied.

$$\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{n1} & A & & \\ \vdots & & \ddots & \\ & & & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & \lambda & \\ & & \lambda \end{bmatrix} \rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{vmatrix}$$

$$\rightarrow \underbrace{\det(A - \lambda I) = p(\lambda) = 0}$$

" the characteristic polynomial " of A.

$$p(\lambda) = 0$$

polynomial of A.

$p(\lambda)=0$

↳ the roots of this polynomial ⇒ eigenvalues of A.

Ex/ $A=\begin{bmatrix}3 & 3\\ 2 & -2\end{bmatrix}$

Find the eigenvalues ✓, eigenvectors, eigenspace, characteristic polynomial ✓ for A.

$\boxed{\det(A-\lambda I)=0}$ ⟶ $\underline{(A-\lambda I)\vec{x}=\vec{0}}$

$$\underbrace{\begin{bmatrix}3 & 3\\ 2 & -2\end{bmatrix}}_{A}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}3-\lambda & 3\\ 2 & -2-\lambda\end{bmatrix}$$

$$\begin{vmatrix}3-\lambda & 3\\ 2 & -2-\lambda\end{vmatrix}=\underbrace{(3-\lambda)(-2-\lambda)}-6=0$$

↳ $\lambda^2+2\lambda-3\lambda-6-6=0$

The characteristic polynomial for A. → $\boxed{\lambda^2-\lambda-12=0}$ ⟶ $(\lambda-4)(\lambda+3)=0$

$\boxed{\lambda_1=4}$ $\boxed{\lambda_2=-3}$

the eigenvalues of A.

For $\lambda_1=4$: $(A-\lambda I)\vec{x}=0$

Find the solutions for this system where $\lambda=4$

$$A-4I=\begin{bmatrix}3 & 2\\ 3 & -2\end{bmatrix}-\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 3 & -6\end{bmatrix}$$

$$\begin{bmatrix}-1 & 2\\ 3 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Solve this system!

$$\xrightarrow{-1r_1\to r_1}\left[\begin{array}{cc|c}1 & -2 & 0\\ 3 & -6 & 0\end{array}\right]\xrightarrow{-3r_1+r_2\to r_2}\left[\begin{array}{cc|c}\overset{x_1}{\textcircled{1}} & \overset{x_2}{-2} & 0\\ 0 & 0 & 0\end{array}\right]$$

$x_2=r\in\mathbb{R}$ free

$x_1=2r$

solution set $=\{(2r,r):r\in\mathbb{R}\}$ → a subspace of $\mathbb{R}^2$ ← $N(A-\lambda I)$

→ eigenspace for $\lambda=4$

$\begin{bmatrix}2r\\ r\end{bmatrix}=r\begin{bmatrix}2\\ 1\end{bmatrix}$

A basis for this set $=\left\{\begin{bmatrix}2\\ 1\end{bmatrix}\right\}$

↳ eigenvector for $\lambda=4$

→ for $\lambda=-3$: $(A-\lambda I)\vec{x}=0$ for $\lambda=-3$

$$\begin{bmatrix}3 & 2\\ 3 & -2\end{bmatrix}-\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}=\begin{bmatrix}6 & 2\\ 3 & 1\end{bmatrix}$$

$$\begin{bmatrix}6 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}6 & 2 & 0\\ 3 & 1 & 0\end{array}\right]\longrightarrow\left[\begin{array}{cc|c}\textcircled{1} & 1/3 & 0\\ 0 & 0 & 0\end{array}\right]$$

$x_2=r\in\mathbb{R}$

$x_1=-1/3r$

$\{(-1/3r,r):r\in\mathbb{R}\}$

eigen space

A basis for this space → $\left\{\begin{bmatrix}-1/3\\ 1\end{bmatrix}\right\}$

$3 \quad 1 \quad | \; 0$ $\qquad$ $0 \quad 0 \quad | \; 0$ $\qquad$ eigenspace

A basis for this space → $\left\{ \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \right\}$ → eigenvector for $\lambda = -3$

Ex

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}_{3\times3}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & 0-\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix}$$

$$(3-\lambda)\begin{vmatrix} -\lambda & -2 \\ -1 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & -2 \\ 2 & -1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$\boxed{(3-\lambda)}\left(\lambda(\lambda+1)-2\right) + \underline{2(-1-\lambda)+4} - \underline{2\left(-2+2\lambda\right)}$$

$\lambda^2+\lambda-2$ $\qquad$ $-2-2\lambda+4 = \underline{2(1-\lambda)}$ $\qquad$ $4-4\lambda = \underline{4(1-\lambda)}$

$(\lambda+2)(\lambda-1)$

$$(1-\lambda)\left[\underline{(\lambda+2)(\lambda-3) + 2 + 4}\right] = 0$$

$\lambda^2+2\lambda-3\lambda-\cancel{6}\cancel{+6}$

$\lambda^2-\lambda$

$$(1-\lambda)\left[\lambda(\lambda-1)\right] = 0 \qquad \boxed{\lambda_1 = 1} \qquad \boxed{\lambda_2 = 0} \rightarrow \text{2 eigenvalues.}$$

For $\lambda_1 = 1$:

$$(A-\lambda I)\vec{x} = 0 \qquad \lambda = 1$$

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}_{3\times3} \qquad \left[\begin{array}{ccc|c} 3-1 & -1 & -2 & 0 \\ 2 & 0-1 & -2 & 0 \\ 2 & -1 & -1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} & x_2 & x_3 & \\ 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = r \in \mathbb{R}$

$x_3 = s \in \mathbb{R}$

$x_1 = (r+2s)/2$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 =$

$x_3 = s \in \mathbb{R}$

$x_1 = (r+2s)/2$

eigenspace $\rightarrow \left\{ \left(\frac{r+2s}{2}, r, s \right) : r, s \in \mathbb{R} \right\} \leq \mathbb{R}^3$

$$\begin{bmatrix} (r+2s)/2 \\ r \\ s \end{bmatrix} = r \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

A basis $= \underbrace{\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{eigenvector corresponding to } \lambda = 1}$

For $\lambda_2 = 0$:

$(A - \lambda I)\vec{x} = 0$ where $\lambda = 0$.

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}_{3\times3} \qquad (A - 0I) = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & -2/3 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow[-2r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1/3 & -2/3 & 0 \\ 0 & 2/3 & -2/3 & 0 \\ 0 & -1/3 & 1/3 & 0 \end{array}\right]$$

$\frac{2}{7} - 1 \qquad \frac{4}{3} - 1$

$$\left[\begin{array}{ccc|c} 1 & -1/3 & \overset{x_3}{-2/3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = r \in \mathbb{R}$ $\qquad x_1 - r/3 - 2r/3 = 0 \Rightarrow x_1 = r$

$\Rightarrow x_2 = r$

$\{ (r, r, r) : r \in \mathbb{R} \} \leq \mathbb{R}^3$

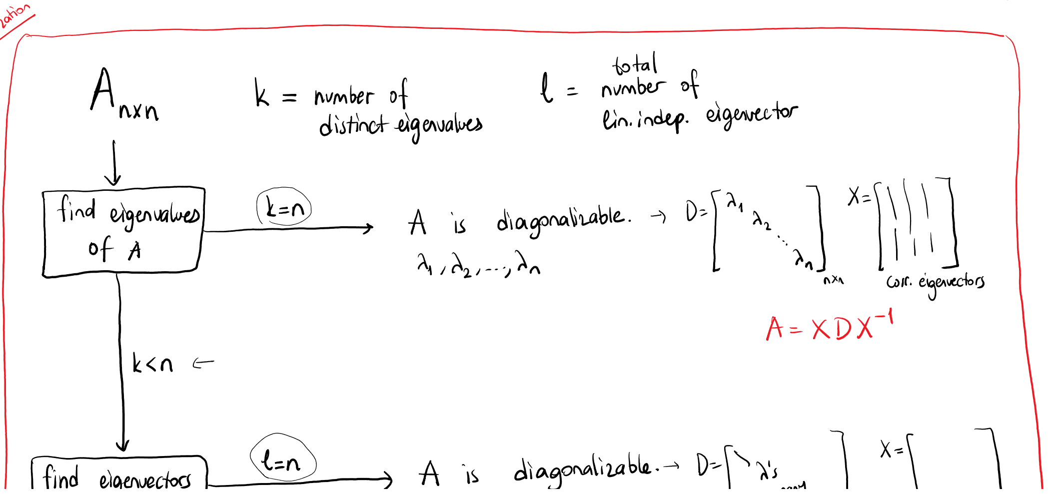
$r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

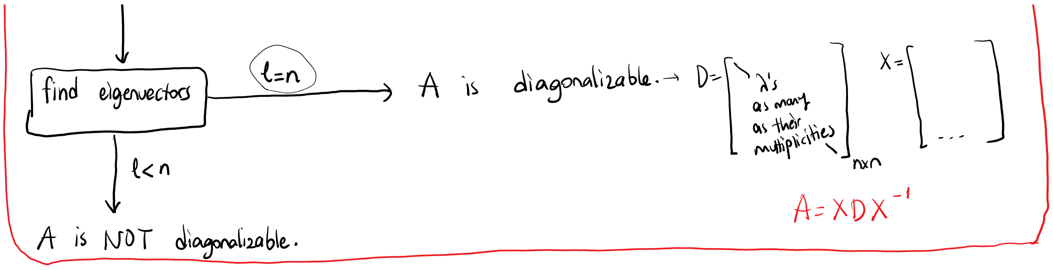
A basis for this space $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\downarrow$

the eigenvector corresponding $\lambda = 0$.

Diagonalization





Ex/

$A=\begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$

a) How many distinct eigenvalues are there ☐ Set of λ's. { ☐ }

b) Is A diagonalizable? Yes [✓] No

c) Find a diagonal matrix D for this diagonalization: $D=\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

d) Find a corresponding invertible matrix $X=\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

{ e) Show that $A=XDX^{-1}$ }

$\det(A-\lambda I)=0$

$$\begin{vmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix}=0 \Rightarrow (6-\lambda)(-1-\lambda)-(-12)=0$$

$$\lambda^2-6\lambda+\lambda-6+12=0 \Rightarrow \lambda^2-5\lambda+6=0 \Rightarrow (\lambda-2)(\lambda-3)=0$$

$-2 \quad -3$

$\boxed{\lambda_1=2}$ $\boxed{\lambda_2=3}$

$A=XDX^{-1}$

$D=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$D=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $X=\begin{bmatrix} 4/3 & 1 \\ 1 & 1 \end{bmatrix}$

2 distinct eigenvalues.

$\Rightarrow$ A is diagonalizable.

eigenvectors: $(A-\lambda I)\vec{x}=0$

for $\lambda=2$: $\rightarrow \left[\begin{array}{cc|c} 6-2 & -4 & 0 \\ 3 & -1-2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right]$ $\quad \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}$

$x_2=r\in\mathbb{R}$

$\Rightarrow x_1=r$

$\begin{bmatrix} r \\ r \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ eigenvector

for $\lambda=3$: $\rightarrow \left[\begin{array}{cc|c} 6-3 & -4 & 0 \\ 3 & -1-3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right]$ $\quad \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2=r\in\mathbb{R}$

$x_1=4r/3$

$\begin{bmatrix} 4r/3 \\ r \end{bmatrix} \rightarrow \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} \rightarrow$ eigenvector

$X=\begin{bmatrix} 1 & 4/3 \\ 1 & 1 \end{bmatrix}$

just to check:

$$A=XDX^{-1}=\begin{bmatrix} 1 & 4/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 3 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 4/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -6 & 8 \\ 9 & -9 \end{bmatrix}=\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$$

$-6+\frac{4}{3}\cdot\cancel{9}^{3}$ $\quad 8+\frac{4}{3}\cdot-9$

$\det X = 1-4/3=-1/3 \neq 0$

$$X^{-1}=\begin{bmatrix} 1 & -4/3 \\ -1 & 1 \end{bmatrix}\cdot\frac{1}{-1/3}=\begin{bmatrix} -3 & 4 \\ 3 & -3 \end{bmatrix}$$

Ex/

$$A=\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}_{4\times 4}$$

neither diagonal nor triangular

$\det(A-\lambda I)=0$

$$\begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}=0$$

$$0+0+(2-\lambda)\begin{vmatrix} 3-\lambda & 0 & 0 \\ 4 & 1-\lambda & 0 \end{vmatrix}=0$$

$$0+0+(2-\lambda)\underbrace{\begin{vmatrix}3-\lambda & 0 & 0\\ 4 & 1-\lambda & 0\\ 0 & 0 & 2-\lambda\end{vmatrix}}_{\text{lower triangular } =(3-\lambda)(1-\lambda)(2-\lambda)}=0$$

$$\Rightarrow\ (2-\lambda)(3-\lambda)(1-\lambda)(2-\lambda)=0$$

$\lambda_1=1 \quad \overset{*}{\lambda}=2 \quad \lambda_3=3 \rightarrow$ 3 distinct eigenvalue

2 ↘ a multiple root with mult. = 2

$(A-\lambda I)\vec{x}=0$

$\underline{\lambda_1=1}$:

$$\left[\begin{array}{cccc|c}3-1 & 0 & 0 & 0 & 0\\ 4 & 1-1 & 0 & 0 & 0\\ 0 & 0 & 2-1 & 1 & 0\\ 0 & 0 & 0 & 2-1 & 0\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}2 & 0 & 0 & 0 & 0\\ 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right]$$

$x_1=0$

$x_2=r\in\mathbb{R}$

$\rightarrow x_3+x_4=0 \Rightarrow x_3=0$

$\rightarrow x_4=0$

$$\left\{\begin{bmatrix}0\\ r\\ 0\\ 0\end{bmatrix}: r\in\mathbb{R}\right\} \qquad \begin{bmatrix}0\\ 1\\ 0\\ 0\end{bmatrix}\rightarrow \text{eigenvector}$$

$\underline{\lambda_2=2}$:

$$\left[\begin{array}{cccc|c}3-2 & 0 & 0 & 0 & 0\\ 4 & 1-2 & 0 & 0 & 0\\ 0 & 0 & 2-2 & 1 & 0\\ 0 & 0 & 0 & 2-2 & 0\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 4 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$x_1=0$

$\Rightarrow x_2=0$

$x_3=r\in\mathbb{R}$

$\Rightarrow x_4=0$

$$\left\{\begin{bmatrix}0\\ 0\\ r\\ 0\end{bmatrix}: r\in\mathbb{R}\right\} \qquad \begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix}\rightarrow \text{eigenvector}$$

$\underline{\lambda_3=3}$:

$$\left[\begin{array}{cccc|c}3-3 & 0 & 0 & 0 & 0\\ 4 & 1-3 & 0 & 0 & 0\\ 0 & 0 & 2-3 & 1 & 0\\ 0 & 0 & 0 & 2-3 & 0\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}0 & 0 & 0 & 0 & 0\\ 4 & -2 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & 0\end{array}\right]$$

$x_2=r\in\mathbb{R} \Rightarrow x_1=r/2$

$\rightarrow x_3=0$

$\rightarrow x_4=0$

$$\left\{\begin{bmatrix}r/2\\ r\\ 0\\ 0\end{bmatrix}: r\in\mathbb{R}\right\} \qquad \begin{bmatrix}1/2\\ 1\\ 0\\ 0\end{bmatrix}\rightarrow \text{eigenvector}$$

+ lin ind

3 eigenvector in total

$\Rightarrow$ A is <u>NOT</u> diagonalizable.

EX/

$$A=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -2\\ 2 & -1 & -1\end{bmatrix}$$

$$\begin{vmatrix}3-\lambda & -1 & -2\\ 2 & 0-\lambda & -2\\ 2 & -1 & -1-\lambda\end{vmatrix}=(3-\lambda)\Big[\underbrace{-\lambda(-1-\lambda)-2}_{\lambda^2+\lambda-2}\Big]+1\Big(\underbrace{2(-1-\lambda)+4}_{-2-2\lambda+4}\Big)-2\Big[\underbrace{-2-(-2\lambda)}_{-2+2\lambda}\Big]$$

$(\lambda-1)(\lambda+2) \qquad 2-2\lambda \qquad +4-4\lambda$

$$(3-\lambda)(\lambda-1)(\lambda+2)+2(1-\lambda)+4(1-\lambda)$$

$$=(1-\lambda)\Big[(\lambda-3)(\lambda+2)+6\Big]=(1-\lambda)\Big[\underbrace{\lambda^2-3\lambda+2\lambda}_{\lambda^2-\lambda}\cancel{-6}\cancel{+6}\Big]$$

$$=\lambda(\lambda-1)$$

$$=(1-\lambda)\lambda(\lambda-1)=0 \qquad \overset{*}{\lambda_1}=1 \quad \lambda_2=0$$

multiple root

mult. = 2

2 distinct eigenvalues <3

we should go through the eigenvectors to decide about diagonalization ←

$(A-\lambda I)\vec{x}=0$

$\underline{\lambda_1 = 1}$: $\left[\begin{array}{ccc|c} 3-1 & -1 & -2 & 0 \\ 2 & 0-1 & -2 & 0 \\ 2 & -1 & -1-1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = r \in \mathbb{R}$

$x_3 = s \in \mathbb{R}$

$x_1 = \underline{\frac{r+2s}{2}}$

$\left\{ \begin{bmatrix} \frac{r+2s}{2} \\ r \\ s \end{bmatrix} : r,s \in \mathbb{R} \right\}$

$\rightarrow r \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow$ eigenvectors for $\lambda = 1$

$\underline{\lambda_2 = 0}$: $\left[\begin{array}{ccc|c} 3-0 & -1 & -2 & 0 \\ 2 & 0-0 & -2 & 0 \\ 2 & -1 & -1-0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right]$

$2x_1 - 2x_3 = 0 \Rightarrow x_1 = x_3 = r \in \mathbb{R}$

$\left.\begin{array}{l} 3r - x_2 - 2r = 0 \\ 2r - x_2 - r = 0 \end{array}\right\} \Rightarrow x_2 = r$

$\left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix} : r \in \mathbb{R} \right\}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ eigenvector for $\lambda = 0$

$\lambda_1^* = 1$ $\underline{\lambda_2 = 0}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $X = \begin{bmatrix} 1/2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\lambda = 1$ $\lambda = 0$

3 lin. indep. eigenvectors $\rightarrow$ A is diagonalizable.

$\det X = 1/2\,(-1) - \cancel{1.1} \cancel{+ 1.1} = -1/2 \neq 0$

columns of X are all linearly indep.

$$A = XDX^{-1}$$

Ex/ $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}_{3\times 3}$

$\det(A - \lambda I) = 0$

$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{vmatrix} = (1-\lambda)\left[\underbrace{(3-\lambda)(-1-\lambda)-5}\right] + 0 + 0 = 0$

$\lambda^2 - 3\lambda + \lambda - 3 - 5 = \underline{\lambda^2 - 2\lambda - 8}$

-4 2

$= (1-\lambda)(\lambda-4)(\lambda+2) = 0$

$\lambda_1 = 1$ $\lambda_2 = 4$ $\lambda_3 = -2$

3 distinct eigenvalues $\Rightarrow$ A is diagonalizable.

$\underline{\lambda_1 = 1}$: $\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1/2 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1/2 & 0 \\ 0 & 0 & -9/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = 0$

$x_3 = 0$

$x_1 = r \in \mathbb{R}$

$\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ eigenvector

$\underline{\lambda_2 = 4}$: $\left[\begin{array}{ccc|c} 1-4 & 2 & 1 & 0 \\ 0 & 3-4 & 1 & 0 \\ 0 & 5 & -1-4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = r \in \mathbb{R}$

$x_2 = r$

$-3x_1 + 2r + r = 0$ $x_1 = r$

$\begin{bmatrix} r \\ r \\ r \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ eigenvect

$\underline{\lambda_3 = -2}$: $\left[\begin{array}{ccc|c} 1-(-2) & 2 & 1 & 0 \\ 0 & 3-(-2) & 1 & 0 \\ 0 & 5 & -1-(-2) & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$-1+2$

$x_3 = r \in \mathbb{R}$

$x_2 = -r/5$

$3x_1 + 2 \cdot \frac{-r}{5} + r = 0$ $x_1 = -r/5$

$-\frac{2r}{5} + \frac{-3r}{5}$

$\begin{bmatrix} -r/5 \\ -r/5 \\ r \end{bmatrix} \rightarrow \begin{bmatrix} -1/5 \\ -1/5 \\ 1 \end{bmatrix}$ eigenvects

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ $X = \begin{bmatrix} 1 & 1 & -1/5 \\ 0 & 1 & -1/5 \\ 0 & 1 & 1 \end{bmatrix}$

$$A = XDX^{-1}$$